Roots of cubic equations

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The following theorem was published by Gerolamo Cardano (1501–1576):

Theorem 1 (Cubic roots) *A cubic equation* $\lambda^3 + x\lambda^2 + y\lambda + z = 0$ *has the solutions*

$$\begin{aligned}
\lambda_1 &= -\frac{x}{3} + W - \frac{P}{W} \\
\lambda_2 &= -\frac{x}{3} - \frac{1-\mathrm{i}\sqrt{3}}{2}W + \frac{P}{\frac{1-\mathrm{i}\sqrt{3}}{2}W} \\
\lambda_3 &= -\frac{x}{3} - \frac{1+\mathrm{i}\sqrt{3}}{2}W + \frac{P}{\frac{1+\mathrm{i}\sqrt{3}}{2}W},
\end{aligned}$$

where

$$Q = -\frac{1}{2}\left(\frac{2x^3}{27} - \frac{xy}{3} + z\right), \quad P = -\frac{1}{3}\left(\frac{x^2}{3} - y\right) \quad \textit{and} \quad W = \sqrt[3]{Q + \sqrt{Q^2 + P^3}}.$$

For $W = 0$*, the solutions are*

$$\lambda_1 = -\frac{x}{3} + \sqrt[3]{2Q}, \quad \lambda_2 = -\frac{x}{3} - \frac{1-\mathrm{i}\sqrt{3}}{2}\sqrt[3]{2Q}, \quad \lambda_3 = -\frac{x}{3} - \frac{1+\mathrm{i}\sqrt{3}}{2}\sqrt[3]{2Q}.$$

Proof Substitute $\lambda = \mu - \frac{x}{3}$ to get the standard form

$$\mu^3 + \mu\left(y - \frac{x^2}{3}\right) + \frac{2x^3}{27} - \frac{xy}{3} + z = 0,$$

which can be shortly written as $\mu^3 + p\mu + q = 0$ by setting $p = y - x^2/3$ and $q = 2x^3/27 - xy/3 + z$. Substitute again using $\mu = \nu - \frac{p}{3\nu}$ and multiply by ν^3 to obtain

$$\left(\nu^3\right)^2 + q\nu^3 - \frac{p^3}{27} = 0, \qquad (1)$$

which is quadratic in ν^3. Solving this yields the solutions

$$-\frac{1}{2}q \pm \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3} \quad = \quad Q \pm \sqrt{Q^2 + P^3} \quad = \quad Q \pm S, \tag{2}$$

where $Q = -q/2$, $P = p/3$ and $S = \sqrt{Q^2 + P^3}$. Let $\sqrt[3]{\alpha}$ denote one of the three radicals β with $\beta^3 = \alpha$. The other two radicals are then given by

$$\exp\left(\frac{2\pi \mathrm{i}}{3}\right) \sqrt[3]{\alpha} = -\frac{1}{2}\left(1 - \mathrm{i}\sqrt{3}\right) \sqrt[3]{\alpha} \;\text{ and }\; \exp\left(\frac{2\pi \mathrm{i}}{3}\right)^2 \sqrt[3]{\alpha} = -\frac{1}{2}\left(1 + \mathrm{i}\sqrt{3}\right) \sqrt[3]{\alpha}.$$

Setting $R_1 = -(1 - \mathrm{i}\sqrt{3})/2$, $R_2 = -1(1 + \mathrm{i}\sqrt{3})/2$ and inserting equation 2 back into the original equations yields the six solutions

$$\begin{aligned}
\lambda_1 &= -\frac{x}{3} + \sqrt[3]{Q+S} - \frac{P}{\sqrt[3]{Q+S}} & \lambda_4 &= -\frac{x}{3} + \sqrt[3]{Q-S} - \frac{P}{\sqrt[3]{Q-S}} \\
\lambda_2 &= -\frac{x}{3} + R_1 \sqrt[3]{Q+S} - \frac{P}{R_1 \sqrt[3]{Q+S}} & \lambda_5 &= -\frac{x}{3} + R_1 \sqrt[3]{Q-S} - \frac{P}{R_1 \sqrt[3]{Q-S}} \\
\lambda_3 &= -\frac{x}{3} + R_2 \sqrt[3]{Q+S} - \frac{P}{R_2 \sqrt[3]{Q+S}} & \lambda_6 &= -\frac{x}{3} + R_2 \sqrt[3]{Q-S} - \frac{P}{R_2 \sqrt[3]{Q-S}}.
\end{aligned}$$

Of these, at most three can be distinct due to the fundamental theorem of algebra. Since λ_1, λ_2 and λ_3 can correspond to different points in the complex plane, it is enough to consider these, i. e. one solution of the root $Q \pm S$ suffices.

For $\nu = 0$, the substitution $\mu = \nu - \frac{p}{3\nu}$ is not valid. In this case, $p = 0$ due to equation 1 and $\mu^3 + q = 0$ yields the three solutions

$$\lambda_1 = -\frac{x}{3} + \sqrt[3]{-q} \qquad \lambda_2 = -\frac{x}{3} + R_1 \sqrt[3]{-q} \qquad \lambda_3 = -\frac{x}{3} + R_2 \sqrt[3]{-q}.$$

□